System of differential equations and stability

Given the following system:

$$\begin{cases} x' = (y - 2x - 2)(y + x - 2), \\ y' = (2y + x + 1)(y - x + 2). \end{cases}$$

a) Find the regions where the derivatives are zero.

b) Find the equilibrium points.

c) Characterize 2 equilibrium points.

Solution

(a)

For $x' = 0$:

$x' = 0$ when:

1. $y - 2x - 2 = 0$,
2. $y + x - 2 = 0$.

These correspond to the following lines in the xy-plane:

1. Line 1: $y = 2x + 2$,
2. Line 2: $y = -x + 2$.

For $y' = 0$:

Similarly, $y' = 0$ when:

1. $2y + x + 1 = 0$,
2. $y - x + 2 = 0$.

These are also lines in the xy-plane:

1. Line 3: $y = -\frac{1}{2}x - \frac{1}{2}$,
2. Line 4: $y = x - 2$.

(b)

Equilibrium points occur where $x' = 0$ and $y' = 0$ simultaneously.

Possible combinations:

1. $y - 2x - 2 = 0$ and $2y + x + 1 = 0$,
2. $y - 2x - 2 = 0$ and $y - x + 2 = 0$,
3. $y + x - 2 = 0$ and $2y + x + 1 = 0$,
4. $y + x - 2 = 0$ and $y - x + 2 = 0$.

Solving each case, we find the equilibrium points:

- $(-1, 0)$,
- $(-4, -6)$,
- $(5, -3)$,
- $(2, 0)$.

(c)

To characterize each equilibrium point, we calculate the **Jacobian matrix** at each point and analyze the eigenvalues.

Jacobian of the system

The Jacobian J is given by:

$$J = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix}.$$

Partial derivatives:

- For $x' = (y - 2x - 2)(y + x - 2)$:

$$\frac{\partial x'}{\partial x} = (y - 2x - 2) - 2(y + x - 2), \quad \frac{\partial x'}{\partial y} = (y + x - 2) + (y - 2x - 2).$$

- For $y' = (2y + x + 1)(y - x + 2)$:

$$\frac{\partial y'}{\partial x} = (y - x + 2) + (2y + x + 1), \quad \frac{\partial y'}{\partial y} = 2(y - x + 2) + (2y + x + 1).$$

We calculate for $(-4, -6)$:
Partial derivatives:

$$\frac{\partial x'}{\partial x} = 0 - 2(-12) = 24,$$
$$\frac{\partial x'}{\partial y} = 0 + (-12) = -12,$$
$$\frac{\partial y'}{\partial x} = -(-15) + 0 = 15,$$
$$\frac{\partial y'}{\partial y} = -15 + 2(0) = -15.$$

The Jacobian at $(-4, -6)$ is:

$$J = \begin{pmatrix} 24 & -12 \\ 15 & -15 \end{pmatrix}.$$

We solve $\det(J - \lambda I) = 0$:

$$\det \begin{pmatrix} 24 - \lambda & -12 \\ 15 & -15 - \lambda \end{pmatrix} = (24 - \lambda)(-15 - \lambda) - (-12)(15) = 0.$$

Simplifying:

$$(24 - \lambda)(-15 - \lambda) + 180 = 0.$$

Expanding:

$$-(24 - \lambda)(15 + \lambda) + 180 = 0 \implies -(360 + 24\lambda - 15\lambda - \lambda^2) + 180 = 0.$$

$$-(360 + 9\lambda - \lambda^2) + 180 = 0 \implies -360 - 9\lambda + \lambda^2 + 180 = 0.$$

$$\lambda^2 - 9\lambda - 180 = 0.$$

Solving the quadratic equation:

$$\lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(-180)}}{2} = \frac{9 \pm \sqrt{81 + 720}}{2} = \frac{9 \pm \sqrt{801}}{2}.$$

The eigenvalues are real and of opposite signs. Conclusion: The point $(-4, -6)$ is a saddle point (unstable).

We calculate for $(2, 0)$:

Partial derivatives:

$$\frac{\partial x'}{\partial x} = -6 - 0 = -6,$$
$$\frac{\partial x'}{\partial y} = -6 + 0 = -6,$$
$$\frac{\partial y'}{\partial x} = -3 + 0 = -3,$$
$$\frac{\partial y'}{\partial y} = 3 + 0 = 3.$$

The Jacobian at $(2, 0)$ is:

$$J = \begin{pmatrix} -6 & -6 \\ -3 & 3 \end{pmatrix}.$$

We solve $\det(J - \lambda I) = 0$:

$$\det \begin{pmatrix} -6 - \lambda & -6 \\ -3 & 3 - \lambda \end{pmatrix} = (-6 - \lambda)(3 - \lambda) - (-6)(-3) = 0.$$

$$(-6 - \lambda)(3 - \lambda) - (-6)(-3) = (-6 - \lambda)(3 - \lambda) - 18 = 0.$$

Expanding:

$$(-18 + 6\lambda + 3\lambda - \lambda^2) - 18 = 0 \implies (-18 + 9\lambda - \lambda^2) - 18 = 0.$$

$$-\lambda^2 + 9\lambda - 36 = 0.$$

Rewriting:

$$\lambda^2 - 9\lambda + 36 = 0.$$

Solving the quadratic equation:

$$\lambda = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(36)}}{2} = \frac{9 \pm \sqrt{81 - 144}}{2} = \frac{9 \pm i\sqrt{63}}{2}.$$

The eigenvalues are complex conjugates with a positive real part:

$$\lambda = \frac{9}{2} \pm i\frac{\sqrt{63}}{2}.$$

Conclusion: The point $(2, 0)$ is an unstable spiral (repelling focus).